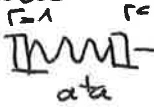
Quantum Optics Lecture 8

Cavity QED

QUANTUM OPTICS OF RESONATORS

Consider interaction of a atom with well isolated (high Q=ω.T) cavity (ie resonator).

$\Gamma=1$ $\Gamma<1$



decay into Free space modes $\sum_k b_k^\dagger b_k$ — cavity ie resonator — atom

Introduce decay: $\underbrace{\int b^\dagger(\omega_k) b(\omega_k) d\omega}$

$$\boxed{\hat H = \sum_{k'} \hbar\omega_{k'} \underbrace{a^\dagger_{k'} a_{k'}}_{\text{cavity mode}} + \sum_k \hbar\omega_k b^\dagger_k b_k + \sum_{kk'} \underbrace{g_{kk'}}_{\text{coupling coefficient}} (\hat a_{k'} \hat b^\dagger_k + \hat b_k \hat a^\dagger_{k'})}$$

For simplicity consider one mode k' of cavity. $\int d\omega\, g_k(\omega)(\hat a^\dagger \hat b(\omega) + h.c.)$

similar treatment as for spontaneous emission: $[\hat b_k, \hat b^\dagger_{k'}] = \delta_{kk'}$ $\wedge$ $[\hat b(\omega), \hat b^\dagger(\omega')] = \delta(\omega-\omega')$

$$\frac{d}{dt}\hat b_k^{(\omega)} = -i\omega \hat b_k^{(\omega)} + g_k \hat a \qquad \cancel{\hat b_k(\omega) = -i\omega \hat b_k(\omega) + g_k \hat a(t)}$$

Thus:

$$\boxed{\hat b_k^{(\omega)} = e^{-i(\omega)(t-t_0)} \hat b_0(\omega) + g_k \cdot \int_{t_0}^{t_1} e^{-i\omega(t-t')} \hat a_k(t') dt'}$$ Variation of Constant

with $t_0 < t$ ie the "forward" i.e noise input are initial conditions.

But for $t < t_1$ the initial conditions are outputs.

$$\boxed{\hat b_k(\omega) = e^{-i\omega(t-t_1)} \hat b_1^{(1)}(\omega) - g_k \int_t^{t_1} e^{-i\omega(t-t')} \hat a(t') dt'}$$

Thus: $\dot{\hat a} = -\frac{i}{\hbar}[\hat a, \hat H_{sys}] - \int_{-\infty}^{\infty} d\omega\, g(\omega)\, \hat b(\omega)$ i.e $\sum_k g_k \to \int d\omega\, g(\omega) \underbrace{\rho(\omega)}_{DOS}$

thus $\underline{g(\omega) = |g_k|^2 \cdot \rho(\omega)}$ Density of states.

$$\boxed{\begin{aligned}\dot{\hat a} &= -i\omega \hat a - \int_{-\infty}^{\infty} d\omega\, g(\omega)\, e^{-i(\omega)(t-t_0)} b_0(\omega) d\omega \\ &\quad - \int_{-\infty}^{\infty} d\omega \underbrace{|g(\omega)|^2}_{|g_k|^2 \rho(\omega)} \int e^{-i(\omega)(t-t')} a(t') dt'\end{aligned}}$$

assume that $|g(\omega)|^2 = |g|^2$ freq independent (1st Markov approx.)

$$\boxed{\int_{-\infty}^{\infty} d\omega\, e^{-i\omega(t-t')} = 2\pi\delta(t-t') \qquad \text{Note: } \int_{t_0}^{t} \delta(t-t') f(t') = \frac{1}{2} f(t) \quad t_0 < t' < t_1}$$ Two key relations necessary

→ Define $\boxed{\dot{\hat a} = -i\omega \hat a - \underbrace{|g|^2 a(t) 2\pi \frac{1}{2}} + \hat a_{in}\sqrt{\kappa}}$

$\kappa = 2\pi |g|^2 \cdot \rho(\omega)$. Decay rate: $-\kappa/2\, a(t)$

<u>input noise</u> $\boxed{\hat a_{in}(t) = \int d\omega\, e^{-i\omega(t-t_0)} b_0(\omega)}$ $\left(\hat= \sum_k e^{-i\omega(t-t_0)} \hat b_k(0)\right)$

$$\boxed{\dot{\hat a} = -i\omega \hat a - \kappa/2\, \hat a(t) + \hat a_{in}\sqrt{\kappa}}$$ (Quantum Langevin Eqns).

Equivalently $\langle \dot{\hat a^\dagger \hat a} \rangle = -\kappa \langle \hat a^\dagger \hat a \rangle + \kappa \langle \hat a^\dagger_{in} \hat a_{in} \rangle$

NOTE $\boxed{\hat a_{out} = -\hat a_{in} + \sqrt{\kappa}\, \hat a}$ INPUT & OUTPUT

NOTE: ATOM in cavity leads to famous Purcell effect.

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Quantum Optics
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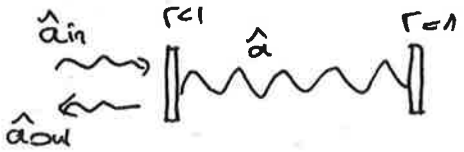
Quantum optics of resonators : input & output relation.
(Ref. Gardiner; Milburn Ch.7)

One can repeat the analysis but now for "backward" time evolution $t_{initial} : t_0$ $t_{final} = t_1$ (initial condition)   and $t < t_1$



repeat analysis for backward case; ie time reversed

Langevin equation:

$$\frac{d}{dt}\hat{a} = -i\omega\hat{a} + \underbrace{|g(\omega)|^2\,\hat{a}(t)\,2\pi}_{+\kappa\,\hat{a}(t)\ \text{sign reversed}} - \underbrace{\sqrt{\kappa}\,\hat{a}_{out}}_{\text{sign reversed}}$$

$$\hat{a}_{out} = \int d\omega\, e^{-i\omega(t-t_1)}$$

Subtracting forward and backward Langevin Eqs yields:

$$\hat{a}_{out} + \hat{a}_{in} = \sqrt{\kappa}\,.\,\hat{a}$$ INPUT & OUTPUT NOD RELATIONS

simple application : TRANSMISSION & REFLECTION PROPERTIES

Transmission thru cavity: (ω_c)

$$d_t\hat{a} = -i\omega_c\hat{a} - \frac{\kappa}{2}\hat{a} + \sqrt{\kappa}\,\hat{a}_{in} \quad \wedge \quad d_t\hat{a}^+ = +i\omega_c - \kappa/2\,\hat{a}^+ + \sqrt{\kappa}\,a_{in}^+$$

Transform to Fourier domain

$$\partial t\begin{pmatrix}\hat{a}_{in}\\ \hat{a}_{in}^+\end{pmatrix} = \begin{pmatrix}-i\omega_c - \kappa/2 & 0\\ 0 & -i\omega_c - \kappa/2\end{pmatrix}\begin{pmatrix}\hat{a}_{in}\\ \hat{a}_{in}^+\end{pmatrix} + \sqrt{\kappa}\begin{pmatrix}\hat{a}_{in}\\ \hat{a}_{in}^+\end{pmatrix}$$

Solve in Fourier domain:

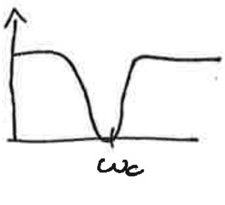
$$\underbrace{i\omega\begin{pmatrix}\hat{a}_{in}(\omega)\\ \hat{a}_{in}^+(\omega)\end{pmatrix}}_{\text{Fourier operators domain}} = \underbrace{\begin{pmatrix}-i\omega_c - \kappa/2 & 0\\ 0 & -i\omega_c - \kappa/2\end{pmatrix}}_{\underline{\underline{M}}}\begin{pmatrix}\hat{a}_{in}\\ \hat{a}_{in}^+\end{pmatrix} + \sqrt{\kappa}\,\overbrace{\underbrace{\begin{pmatrix}\hat{a}_{in}(\omega)\\ \hat{a}_{in}^+(\omega)\end{pmatrix}}_{\vec{a}_{in}(\omega)}}^{\text{Noise inputs.}}$$

$\Rightarrow$ Matrix inversion. $$\sqrt{\kappa}\begin{pmatrix}\hat{a}_{in}(\omega)\\ \hat{a}_{in}^+(\omega)\end{pmatrix} = \left(\underline{\underline{M}} - i\omega\mathbb{1}\right)\begin{pmatrix}\hat{a}\\ \hat{a}^+\end{pmatrix}$$

$$\begin{pmatrix}a(\omega)\\ a^+(\omega)\end{pmatrix} = \frac{1}{\sqrt{\kappa}}\left[\underline{\underline{M}} - i\omega\mathbb{1}\right]^{-1}\begin{pmatrix}\hat{a}_{in}\\ \hat{a}_{in}^+\end{pmatrix}$$

For the transmission / reflection a_{out} this leads to:

$$a_{out}(\omega) = \frac{\kappa/2 + i(\omega_c - \omega)}{\kappa/2 - i(\omega_c - \omega)}\,a_{in}(\omega)$$

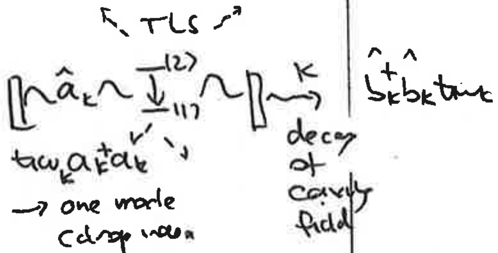


Quantum Optics Lecture 8

Quantum Theory of Light Matter

(Edward)

Purcell Effect in spontaneous emission.



Hamiltonian:

Simplified treatment $\hat{a}_k$ couples to field directly, not freespace modes.

$$H = \sum \hbar\omega \hat{a}^+\hat{a} + \frac{1}{2}\hbar\omega_{12}\hat{\sigma}_z + ug(\sigma^+\hat{a} + \sigma^-\hat{a}^+) + \sum \hbar\omega_k \hat{b}_k^+ b_k + \sum g_k(b_k\hat{a}^+ + b_k^+\hat{a})$$

Approximate treatment (no photons in $|0\rangle_a$ and $|\psi_k\rangle = |0\rangle_1 \otimes \ldots |0\rangle_k$ mod $\hat{b}$) yields: $d_{12} = p_{12}/e$

$$\Gamma_c = \frac{2\pi}{3}\frac{|d_{12}|^2\omega}{2\hbar\varepsilon_0 V}\cdot\left(\frac{2}{\pi}\kappa\right) \quad \propto \quad 2\pi|g|^2 \cdot \underbrace{\rho(\omega)}_{\text{Density of states}} \quad \propto \quad F_{Purcell}\cdot \delta_{FS}\, 2\pi|g|^2$$

Density of states at resonator

$$\rho_{cav}(\omega) = \frac{1}{\pi}\frac{(\kappa/2)}{(\omega-\omega_c)^2 + (\kappa/2)^2}\frac{1}{V}$$

i.e. one mode in Volume V and is spectrum.

⇒ homework.

Note:
- the Purcell effect is an effect that is classical and due to the mode density enhancement.
- Example: electron in a penning trap. (1985 Dehmelt) Inhibited spontaneous emission.

Computation leads to the following Eqs of motion for $\langle\hat{a}^+a\rangle$ and $\langle\sigma_z\rangle$

$$\frac{d}{dt}\langle a^+a\rangle = ig\langle\hat{\sigma}_+\hat{a} - \hat{a}\sigma_-\rangle - \kappa\langle a^+a\rangle + \kappa\underbrace{\bar{n}_{TH}}_{=0}$$

for vacuum environment

$$\frac{d}{dt}\langle\sigma_z\rangle = ig\langle\hat{\sigma}_+ \cancel{\hat{a}} - \cancel{\sigma}_-\hat{a}^+\rangle$$

Note that $\langle\sigma_+a - a\sigma_-\rangle$ depends on $\langle\hat{a}^+\sigma_z a\rangle$ and the equations are not closed ⇒ Fermionic operators (spin)

Simplified treatment for $|0\rangle_r = |0\rangle$.

Quantum Optics
Lecture 8

Topics: – cavity QED basics & spont. emission enhancement
– QND measurements & dispersive regime of cQED.
– cavity input & output relations (theory)
– measurement backaction.

DISPERSIVE REGIME OF CAVITY QED:

(Scully 19.3)

$$\hat{H} = \frac{\hbar\omega_{12}}{2}\hat{\sigma}_z + \hbar\omega a^+ a + \hbar g(\hat{\sigma}^+ a + \hat{\sigma}\hat{a}^+)$$

The Hamiltonian expressed in the $|1\rangle\otimes|n+1\rangle$; $|2\rangle\otimes|n\rangle$ basis:

$$\hat{H} = \begin{pmatrix} -\hbar\omega_{12}/2 + n\hbar\omega & \hbar g\sqrt{n+1} \\ \hbar g\sqrt{n+1} & +\frac{\hbar\omega_{12}}{2} + n\hbar\omega \end{pmatrix}$$

Note that off diagonal terms are equal. $\langle 1, n+1 | \sigma a^+ | 2, n\rangle = \sqrt{n+1}$
$\langle 2, n | \sigma^+ a | 1, n+1\rangle = \sqrt{n+1}$

Matrix can be diagonalized: Eigenstates:

$$|+\rangle_n = \underbrace{\frac{2g\sqrt{n+1}}{(\Omega_n - \Delta) + 4g^2(n+1)}}_{=\cos\Theta_n} \cdot |2, n\rangle - \underbrace{\frac{2g\sqrt{n+1}}{(\Omega - \Delta)^2 + 4g^2(n+1)}}_{=\sin\Theta_n} |1, n+1\rangle$$

$\Omega_n = \sqrt{\Delta^2 + 4g^2(n+1)}$ (ie $\Omega_{n=0} = \sqrt{4g^2} = 2g$ vacuum Rabi splitting)

$$|-\rangle_n = \sin\Theta_n |2, n\rangle + \cos\Theta_n |1, n+1\rangle$$

→ resonant case $\omega_{12} = \omega$
for $\Delta = 0$
this is entangled state
$|\pm, 0\rangle = (|1_A, 1_P\rangle \pm |2_{Atom}, 0\rangle_{Ph})/\sqrt{2}$
VACUUM RABI SPLITTING.

The Energies are given by: (Δ: detuning)

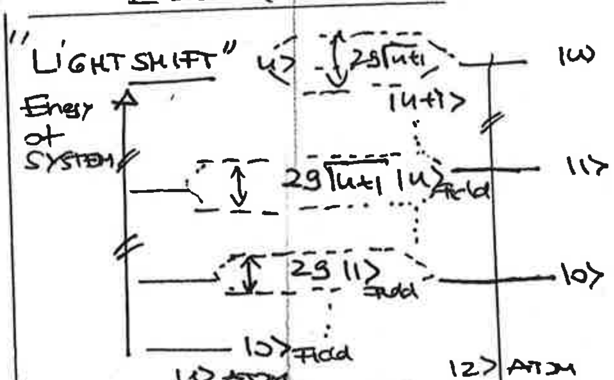
$$E_+ = \hbar(n+1)\omega + \hbar\omega_{12}/2 - \frac{\hbar}{2}(\Omega_n - \Delta)$$
$$E_- = \hbar\omega(n+1) - \hbar\omega_{12}/2 + \underbrace{\frac{\hbar}{2}(\Omega_n - \Delta)}_{\text{Plotted lightshift}}$$

ie $\det(\hat{H} - E^{\pm}\mathbb{1}) \stackrel{!}{=} 0$ ie eigenvalues.

The degeneracy of the two dim. manifold is lifted.

Thus
$\hbar\omega_{12}$
$[\hbar\omega(n+1)] + \hbar\omega_{12}/2$ — E^+_{+n} — $+\hbar/2(\Omega_n - \Delta)$
E^-_{-n} — $-\hbar/2(\Omega_n - \Delta)$
$[\hbar\omega(n+1)] - \hbar\omega_{12}/2$
neglected in plot here.

$\Delta = 0$ $(\omega_{12} = \omega)$



cavity modes coupled to TLS (atom, qubit..)

Graphical presentation of the resonant case of cQED.

very important regime: $\Delta \gg 2g\sqrt{n} = \Omega_n$

Thus $|+\rangle \simeq |1, n+1\rangle$

$|-\rangle \simeq |2, n\rangle$

dispersive regime of CQED: consider ~~...~~

$$\boxed{\begin{aligned} E_{+n} &= \hbar\omega(n-1) + \omega_{12}\hbar/2 - \hbar/2 (\ldots) - \hbar/2(\Omega_n - \Delta) \\ E_{-n} &= \hbar\omega(n) - \omega_{12}\hbar/2 + \hbar/2(\Omega_n - \Delta) \end{aligned}}$$ EXACT diagonalization

yield in the $\Delta \gg g^2$ regime

$$\boxed{\begin{aligned} E_{+n} &= \hbar\omega n + \tfrac{1}{2}\omega_{12}\hbar - \hbar\frac{g^2(n+1)}{\Delta} \\ E_{-n} &= \hbar\omega(n+1) - \tfrac{1}{2}\hbar\omega_{12} + \hbar\frac{g^2(n+1)}{\Delta} \end{aligned}}$$ Total Energy of System in diagonalized state

Since: $\Omega_n \equiv \sqrt{4g^2(n+1)+\Delta^2} \simeq \sqrt{\frac{4g^2(n+1)}{\Delta^2}+1} \simeq 1$ ie $\frac{4g^2(n+1)}{\Delta^2} \ll 1$

$$\frac{\hbar}{2}(\Omega_n - \Delta) \simeq \frac{\hbar}{2}\left[\Delta\left(\sqrt{1+\frac{4g^2(n+1)}{\Delta^2}}\right) - \Delta\right] = \frac{\hbar}{2}\cdot\Delta\frac{4g^2(n+1)}{\Delta^2\,2} = \underbrace{\hbar\cdot\frac{g^2(n+1)}{\Delta}}_{\text{correction to energy}}$$

~~...~~ $\boxed{|+\rangle \simeq |2,n\rangle \qquad |-\rangle \cong |1,n+1\rangle}$

Thus: the shift is $\boxed{\hbar\delta_n = \frac{\hbar g^2(n+1)}{\Delta}}$ Photon number dependent

Note that $E_{1,0} = -\frac{\hbar}{2}\Delta$

The shift per photon is $\left(\frac{\hbar g^2}{\Delta}\right)\cdot n$

KEY POINT: the shift is proportional to photon number (n):

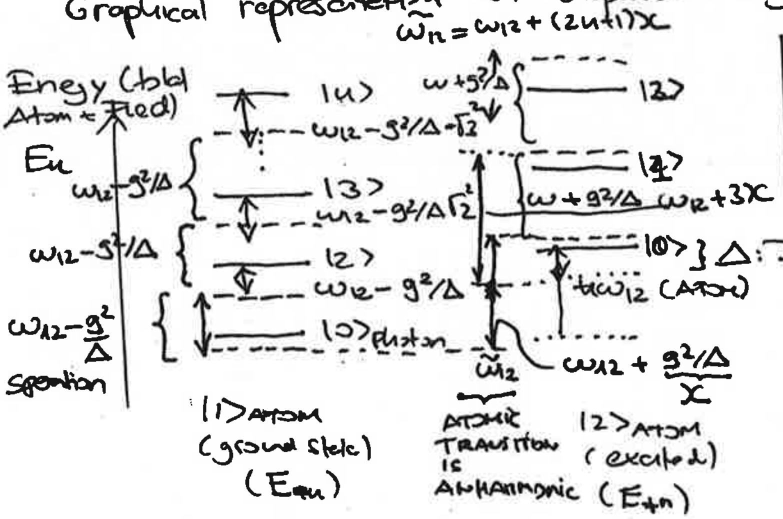
$$\boxed{\hat{H}_{int} = \frac{\hbar g^2}{\Delta}(\hat{a}^+\hat{a})\,\hat{\sigma}_z} \qquad \hat{\sigma}_z = \hat{\sigma}_+\hat{\sigma}_-$$

Thus the frequency of qubit is pulled!

This Hamiltonian has thus special properties. Consider the two level system Energy $\hat{H}_0|1\rangle = -\hbar\omega_{12}/2$ $\hat{H}_0|2\rangle = +\hbar\omega_{12}/2$ (bare freq).

$$\frac{E_2-E_1}{\hbar} \triangleq \omega_{12} = \hbar\omega_{12}\left(1 + k\frac{g^2}{\Delta}\hat{a}^+\hat{a}\right)$$

Graphical representation of dispersive regime. $\chi = g^2/\Delta$



thus the levels are equally spaced by $\omega_{12} \mp g^2/\Delta$ but depend on excited or ground state!

ie $\omega - \omega_{12}$ arises from $|E_{+n} - E_{-n}|$

$E_{+n} = \hbar\omega_{12} + \hbar\omega$

$E_{-n} =$

$$\boxed{E_{\pm,n} = \underbrace{(n+1)\hbar\omega}_{\text{Light}} \pm \frac{\hbar}{2}\sqrt{4g^2(n+1)+\Delta^2}}$$

Exact expression

ie we have a nonlinear pulling of the ~~atomic~~ photon energy (ie frequency) that is NONLINEAR at the SINGLE PHOTON LADDER. or: We have made the OPTICAL FIELD HARMONIC OSCILLATOR ANHARMONIC(!)

re-express states: $|+\rangle \simeq |2,n\rangle$ and $|-\rangle = |1,n+1\rangle$ in basis: (original one ATOM ⊗ FIELD)

$$\boxed{|+,0\rangle \simeq |1,1\rangle - g/\Delta\cdot|2,0\rangle} \qquad \boxed{|-,0\rangle = +\frac{g}{\Delta}|1,1\rangle + |2,0\rangle}$$

~~...~~ admixing of state

Quantum Optics — DISPERSIVE REGIME

– Lecture 8 –

The dispersive regime thus causes a light shift that is state dependent ($|1\rangle$ or $|2\rangle$ in ATOMIC BASIS) $\to$ readout of qubits in a QND manner (more later)

Hence the ATOMIC TRANSITION and PHOTON Transition frequencies are AC STARK / LAMB Shifted.

Photon's: energy shift $\underbrace{E^{\text{phot}}_{\pm,n} \; \pm \hbar\omega n \pm g^2/\Delta}_{\text{ie BOSONIC, equally spaced.}}$ (ie QUBIT / ATOM dependent resonance shift)

Atom frequency shift (ie Transition frequency) $\underline{E_{n\pm} = \pm \frac{\hbar\omega_{12}}{2} + \underbrace{\frac{g^2}{\Delta}n}_{\text{Photon number dependent transition}}}$

The behavior can be reproduced by performing a transformation of the Hamiltonian (unitary)

$$\hat{U} = \exp\left(\frac{g}{\Delta}\,\hat{a}\hat{\sigma}^+ - \hat{a}^+\hat{\sigma}^-\right)$$

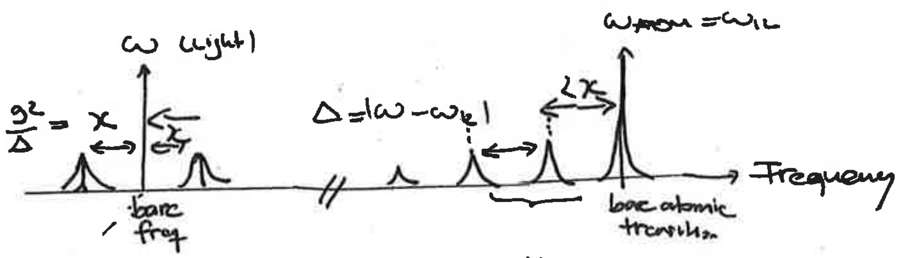
and

$$UHU^+ \underset{\substack{\uparrow \\ \text{2nd order}}}{\approx} \underbrace{\hbar\omega\,\hat{a}^+\hat{a}}_{\text{photon}} + \hbar\omega\frac{g^2}{\Delta}\hat{\sigma}_z + \underbrace{\frac{\hbar}{2}\omega_{12}\hat{\sigma}_z}_{\text{ATOM}} + \underbrace{\frac{g^2}{\Delta}\hat{\sigma}_z}_{\text{shift in level (static)}}$$

regrouping

$$\boxed{\begin{gathered}\hat{H} = \hbar\omega a^+a\underbrace{\left(1 + \frac{g^2}{\Delta}\sigma_z\right)}_{\text{AC Stark shift of photon (ie cavity)}} + \left(\frac{\hbar\omega_{12}}{2} + \frac{g^2}{\Delta}\right)\hat{\sigma}_z \\ \text{or:} \\ \hat{H} = \hbar\omega a^+a + \hat{\sigma}_z\left(\frac{\hbar\omega_{12}}{2} + \frac{g^2}{\Delta} + \underbrace{\frac{g^2}{\Delta}a^+a\,\hbar\omega}_{\text{shift of the ATOM depending on } n}\right)\end{gathered}}$$

Frequency Landscape:



shift is $2\chi n$ for n-photons